Non-Associative Geometry for M5-Brane in C-Field Background

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Dec. 2008

Lie 3-algebra

- $\mathcal{H} = \{ T^a \}$, $[A, B, C] \in \mathcal{H}$
- $[A, B, C]$ = tri-linear, skew-symmetric
- $[A, B, [C, D, E]] = [[A, B, C], D, E] +$

  $[C, [A, B, D], E] + [C, D, [A, B, E]]$

  fundamental id. (generalized Jacobi id.)
- Let $\delta_{(A,B)}C = [A, B, C]$

  $\delta[C, D, E] = [\delta C, D, E] + [C, \delta D, E] + [C, D, \delta E]$

[Filippov '85]

$\mathcal{A}_4$

- *4* generators $\{T_a\}$ $(a = 1,2,3,4)$
- $[T_a, T_b, T_c] = \varepsilon_{abcd}\, T_d$

  [Filippov 85: n-Lie algebras]

- generalization of *su(2)*
- Symmetry transformation is generated by two generators:

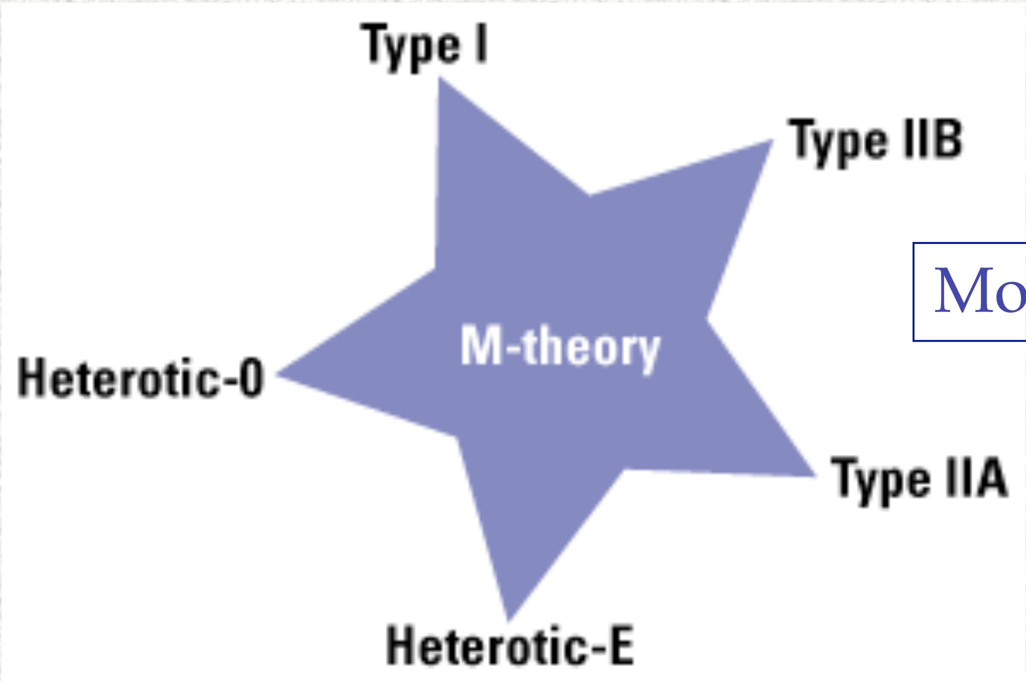
  $\delta A = \Lambda_{ab}\, [T_a, T_b, A]$

  = infinitesimal *SO(4)* rotation

What is M5-brane?

There are 5 superstring theories:

IIA, IIB, I, heterotic $E8 \times E8$, heterotic $SO(32)$



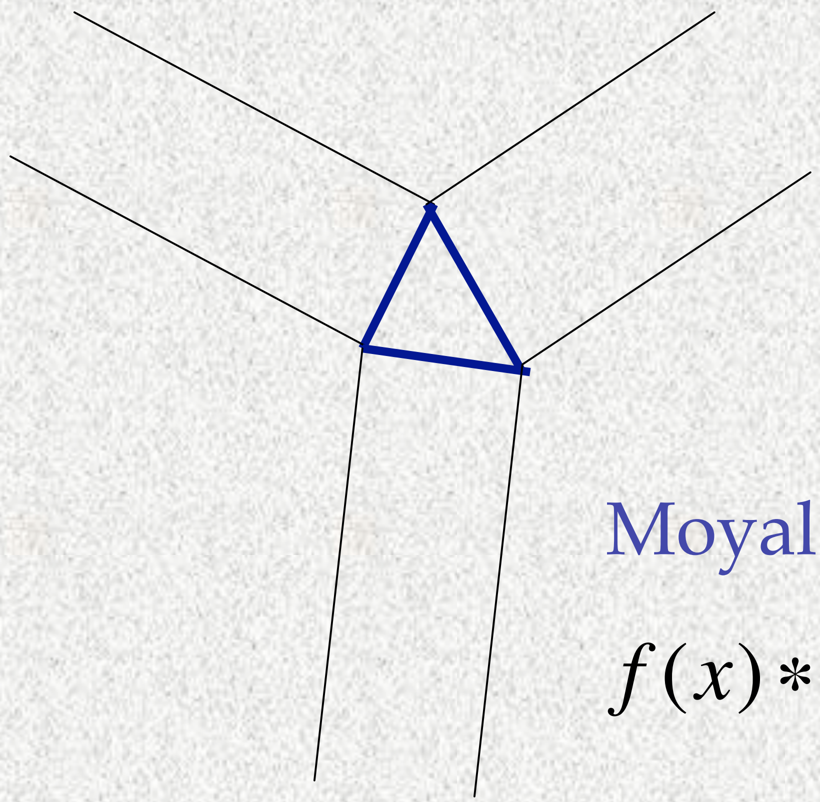
Moduli space of M theory

Non-commutative geometry for D-brane in B-field background

2 ways to see non-commutative geometry:

1. Quantization of open string
   [Chu, Ho: hep-th/9812219]

2. Correlation function
   [Schomerus: hep-th/9903205]



B field = two-form gauge field coupled to string

Moyal product:

$$f(x) * g(x) = e^{i\theta_{ij}\partial_{x^i}\partial_{y^j}} f(x)g(y)\Big|_{x=y}$$

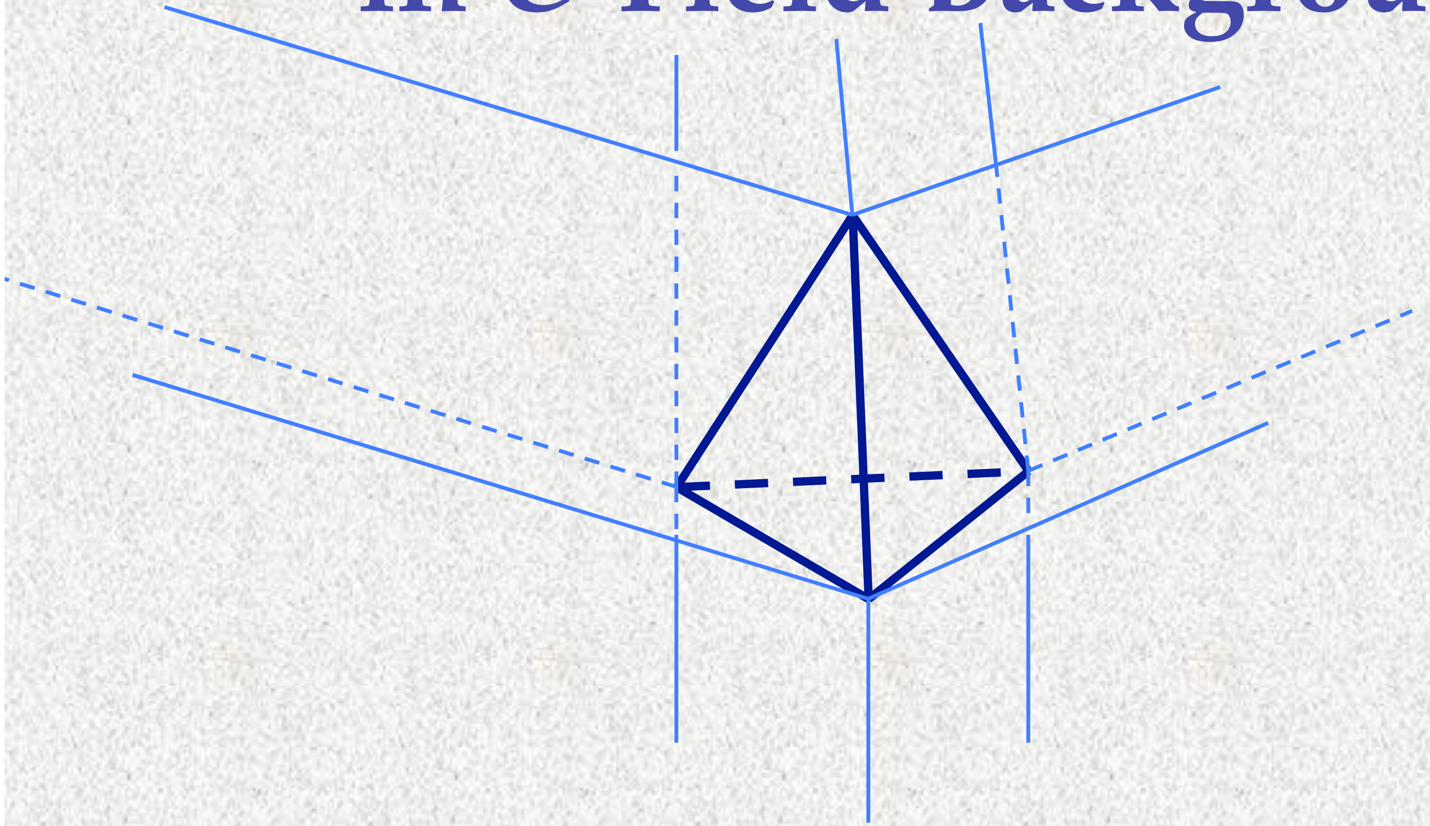
width of string = $B.p$

additional phase = B x Area of triangle

Analogy

- 2-form B field → three-form C field
- open string → open membrane
- Matrix model of multiple D-branes → Bagger-Lambert-Gustavsson model of multiple M2-branes
- NC ~ Poisson ~ Lie bracket ~ commutator
- NA ~ Nambu-Poisson ~ Lie 3-bracket ~ ???

Scattering of Open Membranes in *C*-Field Background



Cross section = $C.p$

Phase = C.volume of tetrahedron [Ho-Matsuo 08]

Nambu Poisson bracket from C-field background

$$(f,g,h) = e^{i\theta_{ijk}\partial_{x^i}\partial_{y^j}\partial_{z^k}} f(x)g(y)h(z)\Big|_{x=y=z}$$

$$[f,g,h] = (f,g,h) - (g,f,h) + \ldots$$

$$[f,g,h] \approx i\theta_{ijk}\,\partial_i f\,\partial_j g\,\partial_k h$$

[Ho-Matsuo 08]

Nambu-Poisson bracket

- Generalization of Poisson brackets [Takhtajan 94]

Recall: Poisson brackets are infinite dim. Lie algs.

- $\{f, g, h\} = P^{abc}\, \partial_a f\, \partial_b g\, \partial_c h$

1. Skew-symmetry
2. Fundamental identity
3. Leibniz rule:

   $\{f, g, h1.h2\} = \{f, g, h1\}h2 + \{f, g, h2\}h1$

- Decomposability theorem:

  Locally $P^{abc} = \varepsilon^{abc}$ for 3 of the n coordinates

Lie 3-algebra and membrane

- string » matrices » Lie algebra
- membrane » ??? » Lie 3-algebra

  [Bagger-Lambert-Gustavsson (BLG) model (07)]

Multiple Membrane Lagrangian [Bagger-Lambert]

$$L = \frac{1}{2}\langle (D_\mu X^I)^2 \rangle - \frac{1}{12}\langle [X^I, X^J, X^K]^2 \rangle + \text{fermions}$$

$$\mu = 0, 1, 2; I, J, K = 3, \cdots, 10.$$

Symmetries of the BLG model

- There is (almost) no free parameter.
- $SO(2,1) \times SO(8)$
- $\mathcal{N} = 8$ SUSY
- Gauge symmetry generated by Lie 3-alg:
  $\delta X = \Lambda_{ab} [T_a, T_b, X]$
  gauge potential $A_{\mu ab}$

M2 to M5

Nambu-Poisson as Lie 3-algebra ⇒

M5-brane in *C*-field background: *(X, B)*

[Ho-Matsuo 08, Ho-Imamura-Matsuo-Shiba 08]

X = transverse coordinates (embedding)

B = 2-form gauge potential with

self-dual *3*-form field strength:

$$H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab}$$

$$H_{abc} = (1/6)\, \varepsilon_{abcdef} H_{def}$$

Self-Dual 3-Form Gauge Field

Gauge transformation:

$$\begin{aligned}\delta A_{\dot\mu\dot\nu}(x,y) &= \partial_{\dot\mu}\Lambda_{\dot\nu}(x,y) - \partial_{\dot\nu}\Lambda_{\dot\mu}(x,y),\\ \delta A_{\mu\dot\mu}(x,y) &= \partial_{\mu}\Lambda_{\dot\mu}(x,y).\end{aligned}$$

Equations of motion:

$$\partial_{\underline{\mu}} F_{\underline{\mu}\dot\mu\dot\nu} = 0,$$

$$\partial_{\dot\nu} F_{\dot\nu\mu\dot\mu} + \partial_{\nu}\tilde{F}_{\nu\mu\dot\mu} = 0. \qquad \Rightarrow \quad F_{\mu\dot\mu\dot\nu} - \tilde{F}_{\mu\dot\mu\dot\nu} = \epsilon_{\dot\mu\dot\nu\dot\lambda}\partial_{\dot\lambda} B_{\mu}.$$

2-form gauge potential:

$$B_{\mu\nu} = -\epsilon_{\mu\nu\lambda} B_{\lambda}, \qquad B_{\mu\dot\mu} = A_{\mu\dot\mu}, \qquad B_{\dot\mu\dot\nu} = A_{\dot\mu\dot\nu}.$$

Self-dual conditions:

$$H_{\mu\nu\lambda} = \frac{1}{6}\epsilon_{\mu\nu\lambda}\epsilon^{\dot\mu\dot\nu\dot\lambda} H_{\dot\mu\dot\nu\dot\lambda}, \qquad H_{\dot\mu\dot\nu\dot\lambda} = \frac{1}{6}\epsilon_{\mu\nu\lambda}\epsilon^{\dot\mu\dot\nu\dot\lambda} H_{\mu\nu\lambda},$$

$$H_{\mu\nu\dot\mu} = -\frac{1}{2}\epsilon_{\mu\nu\lambda}\epsilon^{\dot\mu\dot\nu\dot\lambda} H_{\lambda\dot\nu\dot\lambda}, \qquad H_{\mu\dot\mu\dot\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda}\epsilon^{\dot\mu\dot\nu\dot\lambda} H_{\nu\lambda\dot\lambda}.$$

M5-brane in *C*-field background

$$\begin{aligned}
\mathcal{D}_{\dot\mu}\Phi &\equiv \frac{g^2}{2}\epsilon_{\dot\mu\dot\nu\dot\rho}\{X^{\dot\nu}, X^{\dot\rho}, \Phi\} \\
&= \partial_{\dot\mu}\Phi + g(\partial_{\dot\lambda} b^{\dot\lambda}\partial_{\dot\mu}\Phi - \partial_{\dot\mu} b^{\dot\lambda}\partial_{\dot\lambda}\Phi) + \frac{g^2}{2}\epsilon_{\dot\mu\dot\nu\dot\rho}\{b^{\dot\nu}, b^{\dot\rho}, \Phi\}.
\end{aligned}$$

$$\mathcal{D}_\mu\Phi \equiv D_\mu\Phi = \partial_\mu\Phi - g\{b_{\mu\dot\nu}, y^{\dot\nu}, \Phi\}$$

$$\begin{aligned}
S_X + S_{\rm pot} &= \int d^3x \left\langle -\frac{1}{2}(\mathcal{D}_\mu X^i)^2 - \frac{1}{2}(\mathcal{D}_{\dot\lambda} X^i)^2 - \frac{1}{4}\mathcal{H}^2_{\lambda\dot\mu\dot\nu} - \frac{1}{12}\mathcal{H}^2_{\dot\mu\dot\nu\dot\rho} \right. \\
&\quad \left. -\frac{1}{2g^2} - \frac{g^4}{4}\{X^{\dot\mu}, X^i, X^j\}^2 - \frac{g^4}{12}\{X^i, X^j, X^k\}^2 \right\rangle, \\
S_\Psi + S_{\rm int} &= \int d^3x \left\langle \frac{i}{2}\overline{\Psi}\Gamma^\mu \mathcal{D}_\mu \Psi + \frac{i}{2}\overline{\Psi}\Gamma^{\dot\rho}\Gamma_{\dot1\dot2\dot3}\mathcal{D}_{\dot\rho}\Psi \right. \\
&\quad \left. + \frac{ig^2}{2}\overline{\Psi}\Gamma_{\dot\mu i}\{X^{\dot\mu}, X^i, \Psi\} + \frac{ig^2}{4}\overline{\Psi}\Gamma_{ij}\{X^i, X^j, \Psi\} \right\rangle.
\end{aligned}$$

- Complete nonlinear version of M5-brane theory with self-dual gauge fields

- Gauge symmetry defined by Nambu-Poisson bracket

- Seiberg-Witten map exists.
  [Ho-Imamura-Matsuo-Shiba 08]

Conclusion

- Multiple M5-brane action still mysterious.
- New formulation for self-dual gauge theory.
- Nambu-Poisson structure motivated.
- Non-Associative Geometry?